Mukai flops and deformations of symplectic resolutions

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Abstract

We prove that two projective symplectic resolutions of $\mathbb{C}^{2n}/G$ are connected by Mukai flops in codimension 2 for a finite sub-group $G < \mathrm{Sp}(2n)$. It is also shown that two projective symplectic resolutions of $\mathbb{C}^4/G$ are deformation equivalent.

1 Introduction

A *symplectic variety* is a complex normal variety W with a holomorphic symplectic form on its smooth part which can be extended to a global holomorphic form on any resolution. A resolution $Z \to W$ of W is called *symplectic* if the lifted holomorphic 2-form is again symplectic on Z.

Examples of symplectic varieties include the normalization of a nilpotent orbit closure in a semi-simple complex Lie algebra and the quotient of $\mathbb{C}^{2n}$ by a finite subgroup $G < \mathrm{Sp}(2n)$. The purpose of this paper is to study projective symplectic resolutions of $\mathbb{C}^{2n}/G$.

One way of constructing a symplectic resolution from another is to perform Mukai flops. This process can be described as follows: let W be a symplectic variety and $\pi : Z \to W$ a symplectic resolution. Assume that W contains a smooth closed subvariety Y and that $\pi^{-1}(Y)$ contains a smooth subvariety P such that the restriction of π to P makes P into a $\mathbb{P}^l$-bundle over Y. If $\mathrm{codim}(P) = l$, we can blow up Z along P and then blow down along the other direction, which gives another (proper) symplectic resolution $\pi^+ : Z^+ \to W$. Notice that the resulting variety Z^+ may be not algebraic.

Sometimes one needs to perform simultaneously several Mukai flops to obtain a projective morphism π^+. The diagram $Z \to W \leftarrow Z^+$ is called a *Mukai flop over W with center P*. A *Mukai flop in codimension* 2 is a diagram which becomes a Mukai flop after removing subvarieties of codimension greater than 2.

As to the birational geometry in codimension 2 of projective symplectic resolutions, one has the following conjecture (due to Hu-Yau [HY]):

Conjecture 1 (Hu-Yau). *Any two projective symplectic resolutions of a symplectic variety are connected by Mukai flops in codimension* 2.

This conjecture is true for four-dimensional symplectic varieties by the work of Wierzba and Wiśniewski ([WW]) (partial results had previously been obtained in [BHL], see also [CMSB]) for the existence of flops and by the work of Matsuki [Mat] for the termination of flops. In [Fu], we have verified this conjecture for symplectic resolutions of nilpotent orbit closures. The first result of this note is the following theorem (partial but stronger results had been proven in [Fu]).

Theorem 1.1. *Let $G < \mathrm{Sp}(2n)$ be a finite subgroup. Any two projective symplectic resolutions of $\mathbb{C}^{2n}/G$ are connected by Mukai flops in codimension* 2.

The idea is to reduce the problem to dimension 4, and then apply [WW]. The main technics (see Proposition 2.1 and Lemma 2.4) for this reduction are already contained in [Ka1]. The proofs here are slightly different.

Then we study deformations of symplectic resolutions. Recall that a *deformation* of a variety X (usually not compact) is a flat morphism $\mathcal{X} \xrightarrow{q} S$ from a variety $\mathcal{X}$ to a pointed smooth connected curve $0 \in S$ such that $q^{-1}(0) \simeq X$. A deformation of a proper morphism $X \xrightarrow{f} Y$ is an S-morphism $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ such that $\mathcal{X} \to S$ (resp. $\mathcal{Y} \to S$) is a deformation of X (resp. Y) and $F_0 = f$.

Let $X \xrightarrow{f} Y \xleftarrow{f^+} X^+$ be two proper morphisms. One says that f and f^+ are *deformation equivalent* if there exist deformations of f and f^+: $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xleftarrow{F^+} \mathcal{X}^+$ such that for a general point $s \in S$ the morphisms $\mathcal{X}_s \xrightarrow{F_s} \mathcal{Y}_s \xleftarrow{F_s^+} \mathcal{X}_s^+$ are isomorphisms. Motivated by results of D. Huybrechts ([Huy2]), it is conjectured in [FN] (see also [Ka2]) that

Conjecture 2. *Any two symplectic resolutions of a symplectic variety are deformation equivalent.*

For nilpotent orbit closures of classical type, this conjecture is proved by Y. Namikawa in [Nam] (the case of $\mathfrak{sl}(n)$ had previously been proved in [FN]). In [Ka2], D. Kaledin constructed the so-called *twister deformation* of a symplectic resolution (under mild assumptions). Combining this with a trick of D. Huybrechts ([Huy1]) and results in [WW], we prove the following

Theorem 1.2. *Let $G < \mathrm{Sp}(4)$ be a finite subgroup. Any two projective symplectic resolutions of $\mathbb{C}^4/G$ are deformation equivalent.*

This note ends with a study of symplectic resolutions of the wreath product $W := (\mathbb{C}^2/\Gamma)^{(n)}$, where $\Gamma < \mathrm{SL}(2)$ is a finite subgroup. It is conjectured that any two projective symplectic resolutions of W are connected by Mukai flops with flop center contained in the fiber over $0 \in W$. In the case of Γ being of type A_k and $n = 2$, we give a way to describe all possible projective symplectic resolutions of W.

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2 Birational geometry in codimension 2

We begin with the following proposition, which is proved (as is Lemma 2.4 later) in the formal setting by D. Kaledin ([Ka1] Proposition 5.2).

Proposition 2.1. *Let W be a symplectic variety and Δ^{2l} the open unit disk in $\mathbb{C}^{2l}$. Then any projective symplectic resolution of $W \times \Delta^{2l}$ is of the form $Z \times \Delta^{2l} \xrightarrow{\pi} W \times \Delta^{2l}$, where $Z \xrightarrow{\pi'} W$ is a symplectic resolution and $\pi = \pi' \times \mathrm{id}$.*

Proof. Suppose that we have a symplectic resolution $X \xrightarrow{\pi} W \times \Delta^{2l}$. For any non-zero vector $v \in \Delta^{2l}$, it defines a constant vector field $\mathfrak{t}_v$ on the smooth part, say U of $W \times \Delta^{2l}$. Furthermore, on U, one has an isomorphism of sheaves $\Omega^1 \simeq \mathcal{T}$, under which the vector field $\mathfrak{t}_v$ corresponds to a 1-form α_v. It is easy to show that $\alpha_v = p_2^*\beta$ for some 1-form β on Δ^{2l}, where $p_2 : W \times \Delta^{2l} \to \Delta^{2l}$ is the projection to the second factor. In particular, α_v extends to a well-defined 1-form on the whole of $W \times \Delta^{2l}$. Let $\mathfrak{t}'_v$ be the vector field on X corresponding to the 1-form $\pi^*\alpha_v$ under the isomorphism $\Omega^1_X \simeq \mathcal{T}_X$. Then $\mathfrak{t}'_v$ is the vector field lifting $\mathfrak{t}_v$. Furthermore $\mathfrak{t}'_v$ vanishes nowhere on X, thus it defines a holomorphic flow $\phi_v(t)$ on X (see the proof of Theorem 1.3 [Ka1]).

Let $q : X \to \Delta^{2l}$ be the composition $p_2 \circ \pi$ and $Z = q^{-1}(0)$. Let $\pi' : Z \to W$ be the restriction of π to $Z \to W \times \{0\}$. Then the flow $\phi_v(t)$ satisfies $q(\phi_v(t)(z)) = tv$ for any $z \in Z$. We define a morphism $Z \times \Delta^{2l} \to X$ as follows: $(z, v) \mapsto \phi_v(1)(z)$. One sees easily that this is an isomorphism. Moreover one has $\pi(\phi_v(1)(z)) = (\pi'(z), v)$.

In conclusion, we obtain a decomposition $X = Z \times \Delta^{2l}$, a map $\pi' : Z \to W$ and an isomorphism $\pi = \pi' \times \mathrm{id}$. That π is a symplectic resolution implies that Z is smooth and π' is a symplectic resolution of W. $\square$

The same arguments hold if one replaces Δ^{2l} by $\mathbb{C}^{2l}$. An immediate corollary is the following (which is also proved in [Ka1] Theorem 1.6):

Corollary 2.2. *Let V_i be a symplectic vector space and $G_i < \mathrm{Sp}(V_i)$ a finite subgroup, $i \in \{1, 2\}$. Then $V_1/G_1 \times V_2/G_2$ admits a symplectic resolution if and only if V_1/G_1 and V_2/G_2 both admit symplectic resolutions.*

Proof. Take a smooth point $v \in V_1/G_1$ and a neighborhood isomorphic to the unit disk Δ. If the product admits a symplectic resolution, so does $\Delta \times V_2/G_2$. The precedent proposition then implies that V_2/G_2 admits a symplectic resolution. Similarly V_1/G_1 also admits a symplectic resolution. $\square$

Remark 2.3. We do not know if every projective symplectic resolution of $V_1/G_1 \times V_2/G_2$ is a product of resolutions of V_1/G_1 and V_2/G_2. This is true if G_1 or G_2 is trivial by the precedent proposition.

From now on, let V be a $2n$-dimensional symplectic vector space and $G < \mathrm{Sp}(V)$ a finite subgroup. We denote by W the quotient space V/G. We have the rank stratification on W defined as $V_k = \{v \in V \,|\, \mathrm{codim}\, V^{G_v} = 2k\}$. The quotient $W_k = V_k/G$ is a smooth algebraic variety of dimension $2n - 2k$ and W_0 is the smooth part of W. Moreover, the projection $V_k \to W_k$ is étale (Lemma 4.1 [Ka1]).

Take a component Y of W_k and a connected component V_Y of the preimage of Y in V_k. Let H be the stabilizer of a point in V_Y. Then H is independent of the choice of the point and V_Y is a Zariski open set in the H-fixed subspace V^H. Let $N(H)$ be the normalizer of H in G and $Q(H) = N(H)/H$

the quotient group. One shows that V^H is $N(H)$-invariant. Since H acts trivially on V^H, one obtains an action of $Q(H)$ on V^H, which is a free action on V_Y and we have an isomorphism $V_Y/Q(H) \simeq Y$.

Let V_H be the annihilator of V^H with respect to the symplectic form ω_0 on V. Notice that ω_0 restricted to V^H is again symplectic, thus one has a decomposition $V = V^H \oplus V_H$, which is $N(H)$-invariant. Furthermore $N(H)$ acts symplectically on V_H, i.e., $N(H) < \mathrm{Sp}(V_H)$. This decomposition induces a morphism $\mu : (V^H \times V_H/H)/Q(H) \to V/G$ which maps $V_Y/Q(H) \times \{0\}$ isomorphically to Y and μ is étale in a Zariski open set containing $V_Y/Q(H) \times \{0\}$. For more details, see section 4 of [GK]. This implies (see also Lemma 4.2 [Ka1]):

Lemma 2.4. *Any point in Y admits an analytical open neighborhood which is isomorphic to $\Delta^{2l} \times D_H$, where Δ^{2l} is the unit disk of dimension* $\dim(V^H)$ *and D_H is the image of the unit disk in V_H under the projection $V_H \to V_H/H$.*

Remark 2.5. In the case of $\dim(V_H) = 4$, any symplectic resolution of D_H extends to a symplectic resolution of $S := V_H/H$. In fact, outside the zero point, S has only ADE singularities and $S - \{0\}$ admits a unique symplectic resolution $\tilde{S} \to S - \{0\}$. Now any symplectic resolution of D_H agrees automatically with $\tilde{S}$ over $D_H - \{0\}$, thus it pastes with $\tilde{S}$ to a symplectic resolution of S. If the resolution of D_H is projective, then the one obtained for V_H/H is again projective.

Suppose that we have two projective symplectic resolutions $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$. Let ϕ be the rational map $\pi^{-1} \circ \pi^+ : Z^+ \dashrightarrow Z$.

Lemma 2.6. *The rational map ϕ induces an isomorphism from $(\pi^+)^{-1}(U)$ to $\pi^{-1}(U)$, where $U = W_0 \cup W_1$.*

Proof. By the lemma above, every point $y \in W_1$ admits a neighborhood U_y isomorphic to $\Delta^{2n-2} \times D_H$ for some finite subgroup $H < \mathrm{SL}(2)$. By Proposition 2.1, every symplectic resolution of U_y is a product of Δ^{2n-2} with a symplectic resolution of D_H, while D_H admits a unique symplectic resolution given by the minimal resolution, thus ϕ is an isomorphism from $(\pi^+)^{-1}(U_y)$ to $\pi^{-1}(U_y)$. □

Theorem 2.7. *Two projective symplectic resolutions of W are connected by Mukai flops over W in codimension* 2.

Proof. Let $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$ be two projective symplectic resolutions. By the semi-smallness of symplectic resolutions (Prop. 4.4 [Ka1]), $\pi^{-1}(\overline{W_3})$ (respectively $(\pi^+)^{-1}(\overline{W_3})$) has codimension at least 3 in Z (resp. Z^+). Since we are interested in the codimension 2 birational geometry, we can replace W by $W_0 \cup W_1 \cup W_2$. By the precedent lemma, ϕ is already an isomorphism over $W_0 \cup W_1$.

Take a connected component Y in W_2 and a point $y \in Y$. Then there exists an analytical neighborhood U_y of y isomorphic to $\Delta^{2n-4} \times D_H$ for some finite subgroup $H < \mathrm{Sp}(4)$. By proposition 2.1, the projective symplectic resolution $\pi^{-1}(U_y) \to U_y$ is isomorphic to the product $\Delta^{2n-4} \times X \to U_y$, where $X \to D_H$ is a projective symplectic resolution. Similarly for π^+, one finds another projective symplectic resolution $X^+ \to D_H$. Since Y is connected, X, X^+ and their morphisms to D_H are independent of the choice of y. By Remark 2.5, these two symplectic resolutions come in fact from symplectic resolutions of $\mathbb{C}^4/H$.

By [WW] and [Mat], the birational map $X \dashrightarrow X^+$ is decomposed as a sequence of Mukai flops. Without any loss of generality, one may suppose that $X \dashrightarrow X^+$ is a Mukai flop with flop center $P \subset X$. Since X is independent of the choice of y, one can find a subvariety E in Z which is a fibration over Y with fibers isomorphic to P. By the McKay correspondence (see [Ka3]), irreducible components of codimension 2 in $\pi^{-1}(Y)$ correspond to dimension 2 components in the central fiber of X. The subvariety E is then the irreducible component of codimension 2 in the preimage of Y corresponding to P.

Now if we perform a Mukai flop in Z along E, one obtains another symplectic resolution $X' \xrightarrow{\pi'} W$ such that the rational map $X' \dashrightarrow X^+$ is an isomorphism between preimages of Y.

Now if we do the same operations for other components in W_2, one arrives finally at the resolution π^+. $\square$

We end this section by the following proposition, whose proof is clear.

Proposition 2.8. *Let $W := V/G$ be a quotient symplectic variety. Suppose that for every component Y in W_2, the corresponding 4-dimensional quotient $\mathbb{C}^4/H_Y$ admits a unique projective symplectic resolution. Then any two projective symplectic resolutions of W are isomorphic in codimension 2.*

The following $\mathbb{C}^4/G$ admit a unique projective symplectic resolution:
(i) $\mathbb{C}^2/G_1 \times \mathbb{C}^2/G_2$ where G_1, G_2 are finite subgroups of $\mathrm{SL}(2)$;

(ii) $(T^*\mathbb{C}^2)/G$, where $G < \mathrm{GL}(2)$ such that $\{g | \operatorname{Fix}(g) = 0\}$ form a single conjugacy class.

Case (i) follows from [WW] since the central fiber contains no copies of $\mathbb{P}^2$, while case (ii) is proved in [Fu] (Cor. 1.3).

3 Deformation equivalence

Let V be a $2n$-dimensional symplectic vector space and $G < \mathrm{Sp}(V)$ a finite subgroup. Suppose that we have a projective symplectic resolution $\pi : Z \to W := V/G$. Take a π-ample line bundle L on Z. By [Ka2], there exists a twister deformation of π over the formal disk $\mathrm{Spec}(\mathbb{C}[[x]])$. Since W admits an expanding $\mathbb{C}^*$-action (i.e., positively weighted) which lifts to Z via π, this twister deformation extends to an actual deformation over $S = \mathbb{C}$, say $\mathcal{Z} \xrightarrow{\Phi} \mathcal{W}$ (see Lemma A. 15 and Proposition 5.4 [GK]). Furthermore, for a generic $s \in S$, the morphism $\Phi_s : \mathcal{Z}_z \to \mathcal{W}_s$ is an isomorphism. Moreover, by [Ka2], the Kodaira-Spencer class v of the deformation $\mathcal{Z} \to S$ is nothing but $c_1(L) \in H^1(Z, T_Z) \simeq H^1(Z, \Omega_Z)$.

Let $P \subset Z$ be a subvariety isomorphic to $\mathbb{P}^n$. Denote by $\bar{v}$ the image of the Kodaira-Spencer class v under the morphism $H^1(Z, \Omega_Z) \to H^1(P, \Omega_P)$. The following lemma is a special case of Lemma 3.6 [Huy1]. We omit the proof here.

Lemma 3.1. *If $\bar{v}$ is non-zero, then $\mathcal{N}_{P|\mathcal{Z}} \simeq \mathcal{O}_P(-1)^{\oplus n+1}$.*

Let $p : \tilde{\mathcal{Z}} \to \mathcal{Z}$ be the blow up of $\mathcal{Z}$ along P. Under the assumption of the precedent lemma, the exceptional divisor E is isomorphic to $\mathbb{P}(\mathcal{O}_P(-1)^{\oplus n+1}) = P \times P^*$, where P^* is the dual of P, and the normal bundle $\mathcal{N}_{E|\mathcal{Z}}$ is the tautological bundle. In particular, the restriction of $\mathcal{O}_{\tilde{\mathcal{Z}}}(E))$ to any fiber of $P \times P^* \to P^*$ is $\mathcal{O}(-1)$. By Nakano-Fujiki criterion, there exists a contraction $\tilde{\mathcal{Z}} \to \mathcal{Z}^+$ which blows down E to P^*. Let Z^+ be the Mukai flop of Z along P. Then $\mathcal{Z}^+$ is a one-parameter deformation of Z^+. Let L^+ be the strict transform of L under the rational map $Z \dashrightarrow Z^+$.

Lemma 3.2. *$c_1(L^+)$ is the Kodaira-Spencer class of the deformation $\mathcal{Z}^+ \to S$.*

Proof. Let $U = Z - P$, isomorphic to $U^+ := Z^+ - P^*$. We denote by $v|_U$ (resp. $v^+|_{U^+}$) the image of the Kodaira-Spencer class under the map

$H^1(Z, T_Z) \to H^1(U, T_U)$ (resp. $H^1(Z^+, T_{Z^+}) \to H^1(U^+, T_{U^+})$). Notice that we have an S-isomorphism $\mathcal{Z} - P \simeq \mathcal{Z}^+ - P^*$; thus $v|_U = v^+|_{U^+}$ via the isomorphism $U \simeq U^+$.

The map $H^1(Z, T_Z) \to H^1(U, T_U)$ is injective since $\operatorname{codim}_Z P \geq 2$. Furthermore $v|_U = c_1(L)|_U = c_1(L^+)|_{U^+}$, thus $c_1(L^+)$ is the Kodaira-Spencer class of the deformation $\mathcal{Z}^+ \to S$. □

If furthermore P is mapped to a point by π, then one has another symplectic resolution $Z^+ \to W$ which admits a deformation $\mathcal{Z}^+ \to \mathcal{W}$. The deformations one wants to construct in the following theorem are based on this.

Theorem 3.3. *Let V be a four-dimensional symplectic vector space and $G < \mathrm{Sp}(V)$ a finite subgroup. Then any two projective symplectic resolutions of V/G are deformation equivalent.*

Proof. Let $W = V/G$ and $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$ two projective symplectic resolutions. Take a π^+-ample line bundle L^+ on Z^+. Then we have a deformation of π^+: $\mathcal{Z}^+ \to \mathcal{W}$ such that $c_1(L^+)$ is the Kodaira-Spencer class of $\mathcal{Z}^+ \to S$. Let L be the strict transform to Z of L^+. Then L is π-big. If L is π-nef, then the two resolutions π and π^+ are isomorphic (see [FN] Theorem 2.2).

If L is not π-nef, we can find a $(Z, \epsilon L)$-extremal ray R for small $\epsilon > 0$ (see [KMM]). The locus E of R in Z is contained in $\pi^{-1}(0)$ by Lemma 2.6 and the contraction of R gives a small contraction since $dim(\pi^{-1}(0)) \leq 2$ by the semi-smallness of symplectic resolutions. By [WW], E is a disjoint union of copies isomorphic to $\mathbb{P}^2$. Furthermore L is negative on every curve in E. We can perform a Mukai flop along E to obtain $\pi_1 : Z_1 \to W$. The strict transform L_1 of L is then positive on all curves of E^*. If L_1 is not π_1-nef, then we can continue this process. After finitely many steps, say $Z \dashrightarrow Z_1 \dashrightarrow \cdots \to Z_{l+1}$ one arrives to $\pi_{l+1} = \pi^+$.

Let L_i be the strict transform of L to Z_i and E_i the flop center of $Z_i \dashrightarrow Z_{i+1}$. Then L_{i+1} is positive on curves in E_i^* for $i = 1, \cdots, l$. By Lemma 3.1, the normal bundle $N_{E_l^*|\mathcal{Z}^+}$ is isomorphic to $\mathcal{O}_{E_l^*}(-1)^{\oplus 3}$. Thus we can blow up $\mathcal{Z}^+$ at E_l^* then blow down along the other direction to obtain a deformation of π_l: $\mathcal{Z}_l \to \mathcal{W}$. By Lemma 3.2 and Lemma 3.1, one can perform the same process to E_{l-1}^* in $\mathcal{Z}_l$ and so on. Finally one obtains a deformation of π: $\mathcal{Z} \to \mathcal{W}$. Then the two deformations $\mathcal{Z} \to \mathcal{W} \leftarrow \mathcal{Z}^+$ give the equivalence. □

4 Wreath product and Hilbert schemes

Let $\Gamma < \mathrm{SL}(2)$ be a finite subgroup and $W = (\mathbb{C}^2/\Gamma)^{(n)}$ the n-th symmetric product of $\mathbb{C}^2/\Gamma$. Then W is the quotient of $\mathbb{C}^{2n}$ by the wreath product $\Gamma_n = \Gamma \sim \mathcal{S}_n$. Explicitly, $\Gamma_n = \{(g, \sigma) | g \in \Gamma^n, \sigma \in \mathcal{S}_n\}$ with the multiplication $(g, \sigma) \cdot (h, \tau) = (g\sigma(h), \sigma\tau)$, where $\sigma(h) = (h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n)})$.

Let $S \to \mathbb{C}^2/\Gamma$ be the minimal resolution. Then the composition

$$\mathrm{Hilb}^n(S) \xrightarrow{\tau} S^{(n)} \to (\mathbb{C}^2/\Gamma)^{(n)}$$

gives a projective symplectic resolution $\mathrm{Hilb}^n(S) \xrightarrow{\pi} W$ (see also [Wan]). When Γ is trivial, this is the unique projective symplectic resolution of W (cf. [FN]). However it is not true for a non-trivial Γ. The following problem is open for $n \geq 3$.

Problem 1. *Find out all projective (resp. proper) symplectic resolutions of* $W = \mathbb{C}^{2n}/\Gamma_n$.

Let $C_i, i \in \{1, \ldots, k\}$ be the irreducible components in the exceptional divisor $S \to \mathbb{C}^2/\Gamma$. Then in the central fiber $\pi^{-1}(0)$ there are k disjoint copies of $\mathbb{P}^n$, given by the strict transforms of $C_i^{(n)}$ via τ. In particular, we can perform Mukai flops to obtain many different symplectic resolutions of W. However it is not clear if these resolutions are still projective. An answer to Problem 1 is expected from the following

Conjecture 3. *Any two projective symplectic resolutions of W are connected by a sequence of Mukai flops with flop centers contained in the fiber over $0 \in W$. In particular, they are isomorphic over $W - 0$.*

It is not totally unlikely that the precedent conjecture holds for any quotient variety V/G which is not a product of quotient varieties. For 4-dimensional quotients, this is true thanks to the results in [WW].

A positive answer to this conjecture may imply that Conjecture 2 is valid for projective symplectic resolutions of W, by the arguments of the precedent section and results in [CMSB].

There exists another natural symplectic resolution of W that is constructed as follows (constructed in [Wan]): let $N = |\Gamma|$ be the order of Γ. The action of Γ on $\mathbb{C}^2$ extends to $\mathrm{Hilb}^{nN}(\mathbb{C}^2)$ and $(\mathbb{C}^2)^{(nN)}$. Thus the Hilbert-Chow morphism induces a morphism between Γ-fixed points $\mathrm{Hilb}^{nN,\Gamma}(\mathbb{C}^2) \to (\mathbb{C}^2)^{(nN),\Gamma}$. Notice that $(\mathbb{C}^2)^{(nN),\Gamma}$ is naturally identified with

$W = (\mathbb{C}^2/\Gamma)^{(n)}$. Let $Z_{\Gamma,n}$ be the closure in $\mathrm{Hilb}^{nN,\Gamma}(\mathbb{C}^2)$ of unordered n-tuple of distinct Γ-orbits in $\mathbb{C}^2 - 0$. It is shown in [Wan] that $Z_{\Gamma,n}$ is a connected component of $\mathrm{Hilb}^{nN,\Gamma}(\mathbb{C}^2)$, thus it is smooth and symplectic. Moreover, the morphism $Z_{\Gamma,n} \xrightarrow{\pi^+} (\mathbb{C}^2)^{(nN),\Gamma} \simeq W$ is an isomorphism over W_{reg}, thus it gives a projective symplectic resolution of W.

Problem 2. *Connect the two resolutions π, π^+ by Mukai flops.*

Remark 4.1. π and π^+ are in general non-isomorphic. In the case of $\Gamma = \{\pm 1\}$ and $n = 2$, π and π^+ are the two non-isomorphic projective symplectic resolutions that $(\mathbb{C}^2/\pm 1)^{(2)}$ can have (see [FN], Example 2.7).

In the following we give a way to describe all possible projective symplectic resolutions of $W = (\mathbb{C}^2/\Gamma)^{(2)}$. The irreducible components in $\pi^{-1}(0)$ can be described as follows:

(i) $P_{i,i}$ $(1 \leq i \leq k)$: the strict transform of $C_i^{(2)}$ via τ. They are isomorphic to $\mathbb{P}^2$;

(ii) $P_{i,j}$ $(1 \leq i < j \leq k)$: the strict transform via τ of the image of $C_i \times C_j$ under the morphism $S^2 \to S^{(2)}$. If $C_i \cap C_j = \emptyset$, then $P_{i,j}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. If $C_i \cap C_j = \{x\}$, then $P_{i,j}$ is isomorphic to the one point blow up of $\mathbb{P}^1 \times \mathbb{P}^1$;

(iii) Q_i $(1 \leq i \leq k)$: the preimage $\tau^{-1}(\Delta_{C_i})$, where Δ_{C_i} is the diagonal embedding of C_i in $S^{(2)}$. It is isomorphic to $\mathbb{P}(T_S|_{C_i}) \simeq \mathbb{P}(\mathcal{O}_{C_i}(2) \oplus \mathcal{O}_{C_i}(-2))$, thus a Hirzebruch surface F_4.

Lemma 4.2. *The strict transform of Q_i under any sequence of Mukai flops along components in $\pi^{-1}(0)$ is not isomorphic to $\mathbb{P}^2$.*

Proof. To simplify the presentation, we will only prove the lemma for Γ being of type A_k, i.e., Γ is a cyclic subgroup in $\mathrm{SL}(2)$ of order $k+1$. Let $C_i \cap C_{i+1} = \{x_i\}$ for $i = 1, \cdots, k$. One checks that $l_i := Q_i \cap P_{i,i}$ is a conic in $P_{i,i}$ and a negative section in Q_i. If we perform a Mukai flop along $P_{i,i}$, then l_i is transformed to a conic in $P_{i,i}^*$, which is still called the strict transform of l_i. The strict transform of Q_i is isomorphic to Q_i. Among $P_{i,j}$, only $P_{i-1,i}$ and $P_{i,i+1}$ intersect l_i, both at one point (with multiplicity 2).

One way to make the self-intersection of the strict transform of l_i positive is to flop $P_{i-1,i}$ or $P_{i,i+1}$. To do so, one needs to flop $P_{i,i}$ first. After the flop along $P_{i,i}$, $P'_{i-1,i}$ intersects $P^*_{i,i}$ at one point (which lies on the strict transform of l_i). By this, one sees that the self-intersection of the strict transform of l_i is always negative. $\square$

Thus to construct Mukai flops, one only needs to consider the components $P_{i,j}$. In the following we will assume that Γ is of type A_k (with minor changes, analogue results can be obtained for types D_k, E_l). The configuration of $P_{i,j}$ will be represented in $\mathbb{N}^2$ as follows: $P_{i,j}$ is placed at the position (i,j), represented by a rectangle (resp. an ellipse, a $\oplus$, a circle) if $P_{i,j}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. one point blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, Hirzebruch surface F_1, $\mathbb{P}^2$). These are the vertices of the graph. It is easy to see that the intersection of components of $P_{i,j}$ is either one point or a $\mathbb{P}^1$ if not empty. Two vertices are joined by a solid line (resp. dotted line) if their intersection is a $\mathbb{P}^1$ (resp. a point).

When we perform a Mukai flop at a vertex say $P_{i,j}$, the solid line (resp. dotted line) joining this vertex is replaced by a dotted line (resp. solid line). Other lines are untouched except the following case: the vertex $P_{i,j}$ is joined to two vertices P_1, P_2 by dotted lines. Then after the flop, the two dotted lines are replaced by solid lines, and furthermore P_1 and P_2 are joined by a dotted line. Surely this process is symmetric, i.e., if $P_{i,j}$ is joined to P_1, P_2 by solid lines and P_1, P_2 are joined by dotted line, then after the flop along $P_{i,j}$, the dotted line between P_1 and P_2 should be removed, and the solid lines joining $P_{i,j}$ to P_1, P_2 are replaced by dotted ones.

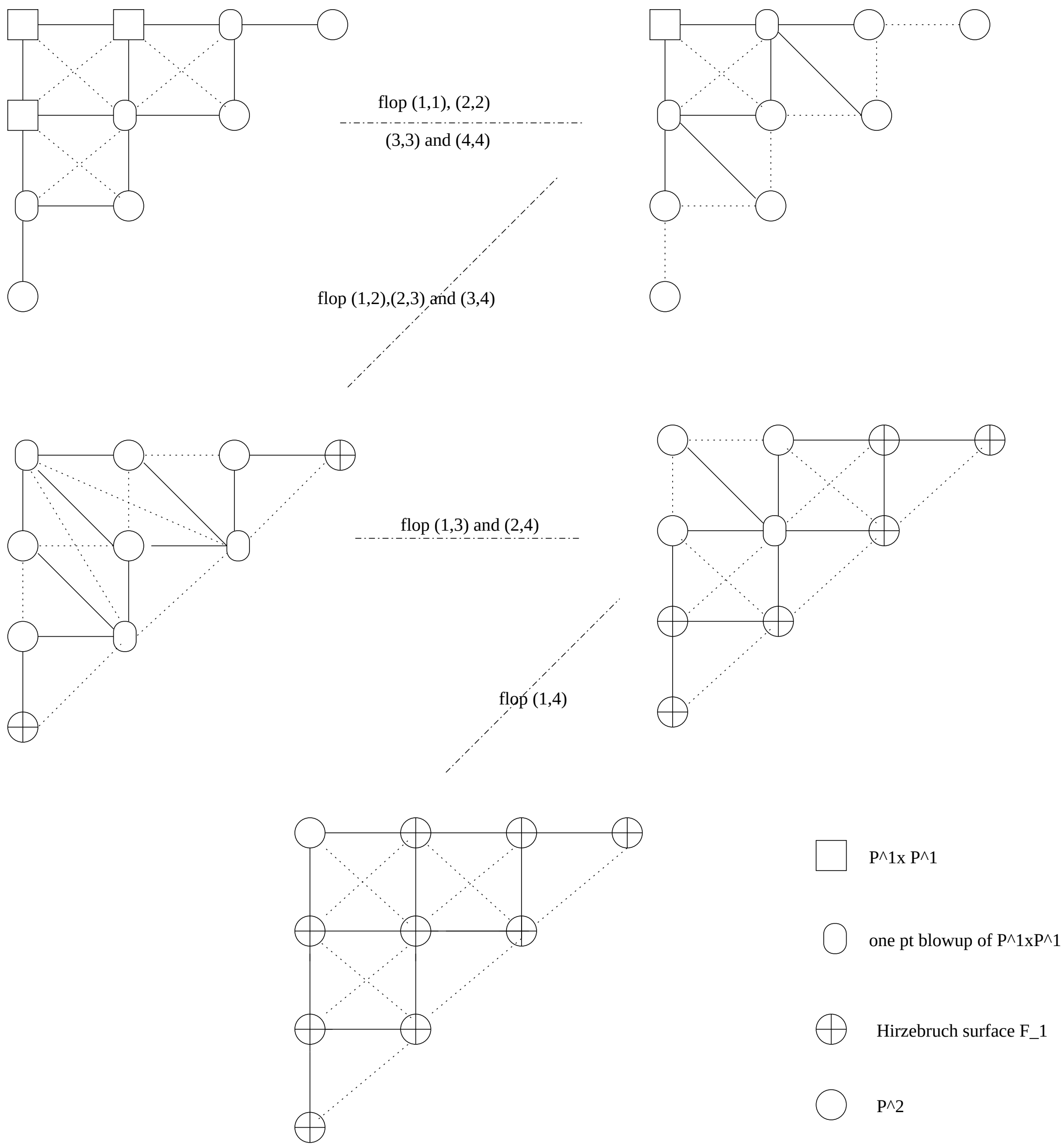
Now we describe how the vertex labels change. Since the process is symmetric, we only describe the changes when P is a vertex joined to $P_{i,j}$ by a solid line. Suppose that P is labeled by an ellipse (i.e., a one point blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$). There are two cases:

(i) the solid line comes from a dotted line, i.e., this line corresponds to the exceptional fiber of the one point blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, then the label at P is changed to a square (i.e., $\mathbb{P}^1 \times \mathbb{P}^1$) ;

(ii) otherwise, the label at P is changed to be $\oplus$ (i.e., F_1).

If P is labeled by a $\oplus$, then it is changed to a circle. The following pictures are examples of symplectic resolutions of $\mathbb{C}^4/\Gamma_2$ with Γ of type A_4.

Any projective symplectic resolutions of W is obtained in this way. However, it is not clear (and it may be not true) that any sequence of Mukai flops gives a projective symplectic resolution. Sometimes one needs to flop simultaneously several disjoint $\mathbb{P}^2$ to obtain a projective resolution.



Example of symplectic resolutions of W, type A_4

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